Nonbondage Number of Graphs and Digraphs: A Survey with Some New Parameters

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Abstract

A set D of vertices in a graph $G = (V, E)$ is a dominating set of G if every vertex in $V – D$ is adjacent to at least one vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set. The nonbondage number $b_n(G)$ of G is the maximum cardinality among all sets of edges $X \subseteq E$ such that $\gamma(G – X) = \gamma(G)$. This concept of the nonbondage number was introduced by Kulli et al. in 1996. Since then, the topic has made some progress and variations. This paper gives a survey on the nonbondage number including known results and problems. Also other types of nonbondage numbers are given.

1. Introduction

All graphs $G = (V, E)$ considered here are finite, undirected without loops and multiple edges. Unless and otherwise stated, the graph considered here have $p = |V|$ vertices and $q = |E|$ edges. Any undefined term in this paper may be found in Kulli [1]. The notations P_p, C_p, K_p, and W_p denote a path, a cycle, a complete graph and a wheel of order p respectively, the notation $K_{m,\ n}$ denotes a complete bipartite graph with $m \leq n$ and $K_{1,\ p-1}$ is a star.

The degree of a vertex v in G is the number of edges of G incident with v and it is denoted by deg v. A vertex of degree zero is called an isolated vertex. An edge incident with a vertex of degree one is called an end edge or a pendant edge. The minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. Let $\lceil x \rceil (\lfloor x \rfloor)$ denote the least (greatest) integer greater (less) than or equal to x. For any vertex $v \in V$, the open neighborhood of v is the set $N(v)=\{u \in V : uv \in E\}$ and the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood $N(S)$ of S is defined by $N(S) = \bigcup_{v \in S} N(v)$, for all $v \in S$ and the closed neighborhood of S is $N[S]=N(S) \cup S$.

Dominating sets were first studied by Berge [2] and Ore [3]. The domination number is an important parameter of graphs which is based upon the dominating set.

A set D of vertices in G is a dominating set if every vertex in $V – D$ is adjacent to at least one vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A γ-set is a minimum dominating set. Recently many new domination parameters are given in the books by Kulli [4, 5, 6].

Among the various applications of the theory of domination that have been considered, the one that is perhaps most often discussed concerns a communication network. Such a network consists of existing communication links between a fixed set of sites. The problem is to select a smallest set of sites at which to place transmitters so that every site in the network that does not have a transmitter is joined by a direct communication link to one that does have a transmitter. This problem reduces to that of finding a minimum dominating set in the graph corresponding to the network. This graph has a vertex representing each site and an edge between two vertices if and only if the corresponding sites have a direct communication link joining them.

To minimize the direct communication links in the network, in [7] Kulli and Janakiram introduced the concept of the nonbondage number as follows:

The nonbondage number $b_n(G)$ of G is the maximum cardinality of all sets of edges $X \subseteq E$ for which $\gamma(G – X) = \gamma(G)$. The precise definition of the nonbondage number is as follows.:

$$b_n(G) = \max\{|X|: X \subseteq E, \gamma(G – X) = \gamma(G)\}.$$

If $\gamma(G – X) = \gamma(G)$ for an edge set X, then X is called the nonbondage set and the maximum one the maximum nonbondage set. If $b_n(G)$ does not exist, we define $b_n(G) = 0$.

The purpose of this paper is to give a survey of results and several new results for graphs and digraphs.

2. Nonbondage Numbers

2.1 Our aim is to determine the nonbondage number for any given graph or communication network. In [8], Kulli determined the exact values of nonbondage numbers for some standard graphs.

Theorem 1 [8]. If P_p is a path with $p \geq 4$ vertices, then

$$b_n(P_p) = \left\lceil \frac{p}{3} \right\rceil - 1.$$

Theorem2 [8]. If C_p is a cycle with $p \geq 3$ vertices, then

$$b_n(C_p) = \left\lceil \frac{p}{3} \right\rceil.$$

Theorem 3 [8]. If K_p is a complete graph with $p \geq 3$ vertices, then

$$b_n(K_p) = \frac{(p-1)(p-2)}{2}.$$

Theorem 4 [8]. If $K_{m,n}$ is a complete bipartite graph with $2 \leq m \leq n$, then

$$b_n(K_{m,n}) = mn - m - n + 2.$$

Theorem 5 [8]. If W_p is a wheel with $p \geq 4$ vertices, then

$$b_n(W_p) = p - 1.$$

Theorem 6. If $K_{1,p}$ is a star , then

$$b_n(K_{1,p}) = 0, \qquad p \geq 1.$$

The domination number of $\gamma(G)$ of a graph G is known. Then the exact value of $b_n(G)$ of G is obtained by Kulli and Janakiram in [7].

Theorem 7[7]. For any graph G, $b_n(G) = q - p + \gamma(G)$.

The exact values of nonbondage numbers for trees and unicyclic graphs are also obtained by Kulli in [8].
Theorem 8 [8]. For any tree T, $b_n(T) = \gamma(T) - 1$.

Theorem 9 [8]. For any unicyclic graph G, $b_n(G) = \gamma(G)$.

2.2 Maximum Nonbondage Sets

Kulli et al. [7] obtained that a necessary and sufficient conditon for an edge to be in every b_n-set of G.

Theorem 10[7]. An edge $e = uv$ is in every b_n-set of G if and only if for every γ-set D of G, $\{u, v\} \subseteq D$ or $\{u,v\} \subseteq V - D$.

An edge e of G is γ-critical if $\gamma(G - e) > \gamma(G)$. If every edge of G is γ-critical, then G is called γ-critical.
Theorem 11[7]. A b_n-set X of G is unique if and only if every edge in $G - X$ is γ-critical.

Theorem 12[7]. For any b_n-set X of G, $G - X$ is γ-critical.

2.3 Bounds

The following lower bound on the nonbondage number is given in terms of its subgraph.
Theorem 13[7]. For any subgraph H of G , $b_n(H) \leq b_n(G)$.
The next result gives a lower bound on $b_n(G)$ in terms of order p.
Theorem 14[7]. If G is a hamiltonian graph, then

$$\left\lceil \frac{p}{3} \right\rceil \leq b_n(G).$$

The following result is another lower bound on $b_n(G)$ in terms of its diameter.
Theorem 15 [7]. For any connected graph G,

$$\frac{\text{diam }(G)-2}{3} \leq b_n(G).$$

The bondage number $b(G)$ of a graph $G = (V, E)$ is the minimum cardinality among all sets of edges $X \subseteq E$ for which $\gamma(G - X) > \gamma(G)$, The idea of the bondage number to given by Bauer et al. in [9]. The concept of the bondage number was introduced by Fink et al. in [10].

The next result relates $b(T)$ and $b_n(T)$.
Theorem 16[7]. Let $T \neq P_4$ be a tree with at least two cutvertices. Then

$$b(T) \leq b_n(T).$$

This bound is attained. For example, if $T=P_5$ or P_6, then $b(T) = b_n(T) = 1$.

The following is an important result which relates the bondage number and nonbondage number of a graph.
Theorem 17[7]. For any graph G,

$$b(G) \leq b_n(G)+1.$$

This bound is attained. For example, $G = P_4$ or C_4. Then $b(P_4)=b_n(P_4)+1 = 2$ and $b(C_4) = b_n(C_4)+1 = 3$.

Problem 1. Characterize trees T for which $b(T) = b_n(T)$.
Problem 2. Characterize graphs G for which $b(G) = b_n(G)+1$.

In [7] Kulli et al. obtained an upper bound involving the maximum degree $\Delta(G)$.
Theorem 18[7]. For any graph G, $b_n(G) \leq 1 - \Delta(G)$.
The following is another lower bound for $b_n(G)$.

Theorem 19[7]. If G is a unicyclic graph and $\gamma(G) = \frac{p}{2}$, then

$$\Delta(G) \leq b_n(G).$$

2.4 Other Bounds

The minimum k such that we can partition $V = S_1 \cup S_2 \cup \ldots \cup S_k$, where each S_i is independent, is the chromatic number $\chi(G)$.
In [11], Brooks established that for any graph G, $\chi(G) \leq \Delta(G)+1$.
Theorem 20 (Kulli[8]). For any graph G,
$b_n(G)+\chi(G) \leq q + 1$.
The complete graphs K_p, $p \geq 3$, achieve this bound.

The vertex connectivity $\kappa(G)$ of G is the minimum number of vertices whose removal from G results in a disconnected graph or trivial graph. The edge connectivity $\lambda(G)$ of G is the minimum number of edges whose removed from G results in a disconnected or trivial graph. The Whitney inequality is stated as $\kappa(G) \leq \lambda(G) \leq \delta(G)$ for any graph G.

Theorem 21 (Kulli [8]). For any connected graph G, $b_n(G)+ \kappa(G) \leq q$.
The cycle C_3 achieves this bound.
Theorem 22 . For any connected graph G,
$b_n(G) + \lambda(G) \leq q$.
The cycle C_3 achieves this bound.

We suggest the following problems for further study.
Problem 3[8]. $b_n(G)+\chi(G) \leq q+1$. Characterize the corresponding extremal graphs.

Problem 4[8]. $b_n(G)+\kappa(G) \leq q$. Characterize the corresponding extremal graphs.

Problem 5. $b_n(G)+\lambda(G) \leq q$. Characterize the corresponding extremal graphs.

2.5 Northaus Gaddum Type Results

Kulli [8] obtained the following Nordhaus-Gaddum type inequalities concerning the nonbondage number of a graph and its complement.
Theorem 23 (Kulli [8]). For a graph G and its complement $\bar{G}$,

$$b_n(G)+b_n(\bar{G}) \leq \frac{(p-1)(p-2)}{2}.$$

Theorem 24 (Kulli [8]). If G and $\bar{G}$ are connected, then

$$b_n(G)+b_n(\bar{G}) \leq \frac{p(p-3)}{2}.$$

Theorem 25 (Kulli [8]). If T and $\bar{T}$ are trees, then

$b_n(T)+b_n(\overline{T})\le p-2.$

3. Other Nonbondage Numbers

Many domination parameters are obtained by combining domination with another graph theoretical property. So one can define a new nonbondage number as long as a variation of domination number is given. Now we survey results on the nonbondage numbers of such domination parameters.

3.1 Total Nonbondage Numbers

A domination set D of a graph G without isolated vertices is a total dominating set if the induced subgraph $\langle D\rangle$ has no isolated vertices. The total domination number $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set. It is clear that $\gamma(G) \le \gamma_t(G) \le 2\gamma(G)$. This concept was introduced by Cockayne et al in [12]. Still total domination in graphs has been extensively studied in the literature. In [13], Henning gave a survey of selected recent results on total domination in graphs.

The total nonbondage number $b_{tn}(G)$ of G without isolated vertices is the maximum cardinality among all sets of edges $X \subseteq E$ such that $G - X$ has no isolated vertices and $\gamma_t(G - X) = \gamma_t(G)$.

Clearly $b_{tn}(G)$ may not exist for some graphs. For example $b_{tn}(K_{1,p})$ does not exist.

The total nonbondage number of a graph was introduced by Kulli in [14].

The exact values of $b_{tn}(G)$ for some standard graphs are obtained by Kulli [14].

Theorem 26 (Kulli [14]).

$b_{tn}(P_p)=\left\lfloor \frac{p}{3}\right\rfloor-1,$ for $p\ge 6$.

$b_{tn}(C_p)=\left\lfloor \frac{p}{3}\right\rfloor,$ for $p\ge 3$.

$b_{tn}(K_{m,n})=mn-m-n+1,$ for $2 \le m \le n$.

$b_{tn}(W_p)=p-1,$ for $p\ge 4$.

Problem 6. Determine the total nonbondage number of certesian product $P_m \times P_n$ of two paths P_m and P_n .

Problem 7. Determine the total nonbondage number of the cartesian product $C_m \times P_2$ of cycle C_m and path P_2.

Problem 8. Determine the total nonbondage number of the cartesian product $C_m \times C_n$ of two cycles C_m and C_n.

The exact values of $b_{tn}(G)$ is obtained in terms of the order and the size of G by Kulli in [14].

Theorem 27 (Kulli [14]). If G is a graph without isolated vertices and $\Delta(G) = p - 1$, then

$b_{tn}(G) = q - p + 1$.

Kulli gave an upper bond on $b_{tn}(G)$ in terms of the maximum degree $\Delta(G)$ and the size of G.

Theorem 28 (Kulli[14]). If G is a graph without isolated vertices, then

$b_{tn}(G)\le q-\Delta(G).$

In [14], Kulli established a relation between the total nonbondage number $b_{tn}(G)$ and the chromatic number $\chi(G)$.

Theorem 29 (Kulli [14]). For any graph G without isolated vertices,

$b_{tn}(G)+\chi(G)\le q+1.$

The graphs K_p, $p \ge 3$, achieve this bound.

Kulli obtained a relation between the total nonbondage number and the connectivity.

Theorem 30 (Kulli [14]). For any graph G without isolated vertices,

$b_{tn}(G)+\kappa(G)\le q.$

The cycle C_3 achieves this bound.

Theorem 31. For any graph G without isolated vertices,

$b_{tn}(G)+\lambda(G)\le q.$

Problem 9. Characterize graphs G for which $b_{tn}(G)+ \chi(G) \le q+1$.

Problem 10. Characterize graphs G for which $b_{tn}(G)+\kappa(G)=q$.

Problem 11. Characterize graphs G for which $b_{tn}(G)+\lambda(G)=q$.

A Nortdhaus Gaddum type result for $b_{tn}(G)$ is given in [14].

Theorem 32. For graphs G and $\overline{G}$ without isolated vertices,

$b_{tn}(G)+b_{tn}(\overline{G})\le \frac{(p-1)(p-2)}{2}.$

The total bondage number $b_t(G)$ of G is the minimum cardinality of a subset $X \subseteq E$ with the property that $G - X$ contains no isolated vertices and $\gamma_t(G-X)>\gamma_t(G)$. This concept was introduced by Kulli and Patwari in [15].

We now consider a relation between $b_{tn}(G)$ and $b_t(G)$.

Theorem 33 (Kulli [14]). For any graph G without isolated vertices,

$b_t(G)\le b_{tn}(G)+1.$

3.2 Strong Nonbondage Numbers

A set $D \subseteq V$ is a strong dominating set if every vertex in $V–D$ has a neighbour u in D such that the degree of u is not smaller than the degree of v. The strong domination number $\gamma_s(G)$ is the minimum cardinality of a strong dominating set. The strong domination in graphs was introduced by Sampathkumar et al. in [16].

The strong nonbondage number $b_{sn}(G)$ of a nonempty graph G is the maximum cardinality among all sets of edges $X\subseteq E$ such that $\gamma_s(G-X)=\gamma_s(G)$.

In [17] Ebadi et al. studied the strong nonbondage number of a graph and calculated the exact values of $b_{sn}(G)$ for some standard graphs.

Theorem 34 [17].

i) For any path P_p with $p\ge 3$ vertices,

$b_{sn}(P_p)=\left\lceil \frac{p-1}{3}\right\rceil.$

ii) For any cycle C_p with $p\ge 3$ vertices,

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$b_{sn}(C_p) = \left\lceil \frac{p}{3} \right\rceil.$

iii) For any complete graph K_p with $p \geq 2$ vertices,

$b_{sn}(K_p) = \frac{(p-1)(p-2)}{2}.$

iv) For any complete bipartite graph $K_{m,n}$ with $2 \leq m \leq n$,

$$b_{sn}(K_{m,n}) = mn - m - n + 2, \quad \text{if } m = n,$$
$$= mn - n, \quad \text{if } m < n,$$

v) For any wheel W_p with $p \geq 4$ vertices,

$b_{sn}(W_p) = p - 1.$

The exact value of $b_{sn}(G)$ is established in [17] .

Theorem 35 [17]. For any graph G,

$b_{sn}(G) = q - p + b_s(G).$

Theorem 36 [17]. For any graph G,

$b_n(G) \leq b_{sn}(G).$

Theorem 37 [17]. For any graph G,

$q - p \leq b_{sn}(G) \leq q - \Delta(G).$

Theorem 38 [17]. If H is a subgraph of G, then

$b_{sn}(H) \leq b_{sn}(G).$

The following result is a lower bound on $b_{sn}(G)$ in terms of its diameter.

Theorem 39[17]. For any connected graph G,

$\frac{\text{diam }(G) - 2}{3} \leq b_{sn}(G).$

The following result is another lower bound on $b_{sn}(G)$.

Theorem 40 [17]. If G is a Hamiltonian graph, then

$\left\lceil \frac{p}{3} \right\rceil \leq b_{sn}(G).$

The following result is an upper bound on $b_{sn}(T)$.

Theorem 41[17]. For any tree T,

$b_{sn}(T) \leq \left\lceil \frac{4(p-2)}{7} \right\rceil.$

The strong bondage number $b_s(G)$ of G is the minimum cardinality among all sets of edges $X \subseteq E$ such that $\gamma_s(G - X) > \gamma_s(G)$. This concept was introduced by Ghoshal et al. in [18].

The following result relates between $b_s(G)$ and $b_{sn}(G)$.

Theorem 42 [17]. For any graph G,

$b_s(G) \leq b_{sn}(G) + 1.$

A lower bound for $b_{sn}(G)$ is given in [17].

Theorem 43[17]. If G is a unicyclic graph and $\gamma_s(G) = \frac{p}{2}$, then

$\Delta(G) \leq b_{sn}(G).$

Ebadi et al. [17] established bounds on $b_{sn}(G)$ containing cycles in terms of its girth. The girth $g(G)$ of G is the length of a smallest cycle in G.

Theorem 44 [17]. If G a nontrivial graph of order $p \geq 3$ and $g(G) \geq 3$, then

$b_{sn}(G) \geq \left\lceil \frac{g(G)}{3} \right\rceil.$

Theorem 45[17]. If G is a connected graph of order p and $g(G) \geq 3$, then

$b_{sn}(G) \leq q - \left\lceil \frac{2g(G)}{3} \right\rceil.$

They also established the following result.

Theorem 46[17]. An edge $e = uv$ is in every b_{sn}-set of G if and only if for every γ_s-set D of G,

(i) $\{u, v\} \subseteq D$,

(ii) $\{u,v\} \subseteq V - D$,

(iii) $u \in D$ and $v \in V - D$ implies $|N(v) \cap D| \geq 2$.

Kulli (8) obtained another upper bound for $b(G)$ involving the strong nonbondage number $b_{sn}(G)$.

Theorem 47[8]. For any graph G,

$b(G) \leq b_{sn}(G) + 1.$

Kulli [8] established the following inequalities.

Theorem 48[8]. For any graph G,

$b_{sn}(G) + \chi(G) \leq q + 1.$

The path P_3 achieves this bound.

Theorem 49[8]. If G is a unicyclic graph and $\gamma(G) = \frac{p}{2}$, then

$\chi(G) \leq b_{sn}(G) + 1.$

Theorem 50[8]. For any graph G,

$b_{sn}(G) + \kappa(G) \leq q.$

Theorem 51. For any graph G,

$b_{sn}(G) + \lambda(G) \leq q.$

Problem 12. Characterize graphs G for which

$b_{sn}(G) + \chi(G) = q + 1.$

Problem 13. Characterize graphs G for which

$b_{sn}(G) + \kappa(G) = q.$

Problem 14. Characterize graphs G for which

$b_{sn}(G) + \lambda(G) = q.$

A Nordhaus - Gaddum type result is obtained in [17].

Theorem 52[17]. For a graph G and its complement $\overline{G}$,

$b_{sn}(G) + b_{sn}(\overline{G}) \leq \frac{(p-1)(p-1)}{2}.$

3.3 Efficient Nonbondage Numbers

A set D of vertices in a graph G is an efficient dominating set if every vertex in $V – D$ is adjacent to exactly one vertex in D. The efficient domination number $\gamma_e(G)$ of G is the minimum cardinality of an efficient dominating set of G. This concept was introduced by Cockayne et al in [19].

It is clear from the definition that an efficient dominating set is certainly an independent set.

In [20], Kulli introduced the efficient nonbondage number as follows.

The efficient nonbondage number $b_{en}(G)$ of G is the maximum cardinality among all sets of edges $X\subseteq E$ such that $G – X$ has no isolated vertices and $\gamma_e(G-X)=\gamma_e(G)$.

In [20], Kulli obtained the exact values of $b_{en}(G)$ for some standard graphs.

Theorem 53 (Kulli [20]).

$b_{en}(P_p)=\left\lceil \frac{p-1}{3}\right\rceil$, for $p \geq 4$.

$b_{en}(C_p)=\left\lceil \frac{p}{3}\right\rceil$, for $p \geq 3$.

$b_{en}(K_p)=\frac{(p-1)(p-2)}{2}$, for $p \geq 3$.

$b_{en}(K_{m,n})=mn-m-n+2$, for $2 \leq m \leq n$.

$b_{en}(W_p)=p-1$. for $p \geq 4$.

Exact value of the efficient nonbondage number of a graph G is also established in [20].

Theorem 54[20]. For any graph G without isolated vertices,
$b_{en}(G)=q-p+\gamma_e(G)$.

Theorem 55[20]. For any graph G without isolated vertices and $\Delta(G)=p-1$,
$b_{en}(G)=q-p+1$.

An upper bound on $b_{en}(G)$ in terms of the maximum degree $\Delta(G)$ and the size q of G is given in [20].

Theorem 56 [20]. If G is a graph without isolated vertices, then
$b_{en}(G)\leq q-\Delta(G)$.

A relation between $b_{en}(G)$ and $\chi(G)$ is established in [20].

Theorem 57. If G is a graph without isolated vertices., then
$b_{en}(G)+\chi(G))\leq q+1$.

The cycle C_3 achieves this bound.

The following result is obtained in [20].

Theorem 58. If G is a graph without isolated vertices, then
$b_{en}(G)+\kappa(G))\leq q$.

The cycle C_3 achieves this bound.

Theorem 59. If G is a graph without isolated vertices, then
$b_{en}(G)+\lambda(G))\leq q$.

The cycle C_3 achieves this bound.

The efficient bondage number $b_e(G)$ of G without isolated vertices is the minimum cardinality among all sets of edges $X\subseteq$E for which $\gamma_e(G-X)>\gamma_e(G)$.

This concept was introduced by Kulli and Soner in [21].

The following result gives an upper bound for $b_e(G)$.

Theorem 60 (Kulli [20]). For any graph G without isolated vertices,
$b_e(G)\leq b_{en}(G)+1$.

3.4 Restrained Nonbondage Numbers (or Cototal Nonbondage Numbers)

A dominating set S of G is a restrained dominating set if the induced subgraph $\langle V – S\rangle$ contains no isolated vertices. The restrained domination number $\gamma_r(G)$ of G is the minimum cardinality of a restrained dominating set of G. This concept was defined by Telle and Proskurowski in [22] and was also defined independently as cototal domination in graphs by Kulli et al. in [23]

Clearly $\gamma_r(G)$ exists and $\gamma(G) \leq \gamma_r(G)$ for any nonempty graph G.

Recently Kulli [24] introduced the concept of the restrained nonbondage number as follows.

The restrained nonbondage number $b_{rn}(G)$ of a nonempty graph G is the maximum cardinality among all sets of edges $X\subseteq E$ such that $\gamma_r(G – X) = \gamma_r(G)$.

The exact values of $b_{rn}(G)$ for some special graphs are obtained.

Theorem 61. If P_p is a path with $p\geq 2$ vertices, then
$b_{rn}(P_p) = 0$.

Theorem 62. If C_p is a cycle with $p\geq 3$ vertices, then
$b_{rn}(C_p) = 1$, if $p = 1, 2 \pmod 3$,
$= 0$, otherwise.

Theorem 63. If K_p is a complete graph with $p\geq 3$ vertices, then
$b_{rn}(K_p)=\frac{(p-1)(p-3)}{2}$, if $p=2k-1$, $k\geq 2$,
$=\frac{p(p-4)}{2}+1$ if $p=2k-2$, $k\geq 3$.

Theorem 64. If $K_{m,n}$ is a complete bipartite graph, then
$b_{rn}(K_{m,n})=mn-2m-2n+5$, for $2 \leq m \leq n$.

Theorem 65. If W_p is a wheel with $p \geq 4$ vertices, then
$b_{rn}(W_p)=1$, if $p = 4$
$=k+1$, if $p=2k+3$, $p=2k+4$, $k \geq 1$.

The restrained bondage number $b_r(G)$ of a nonempty graph G is the minimum cardinality among all sets of edges $X\subseteq E$ for which $\gamma_r(G-X)>\gamma_s(G)$. This concept was introduced by Hattingh and Plummer in [25].

A relation between the restrained bondage number and restrained nonbondage number of a graph is established.

Theorem 66. For any nonempty graph G,

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$b_r(G) \le b_{rn}(G)+1.$

For p = 1, 2 (mod 3), the cycles C_p achieve this bound.

3.5. Total Restrained Nonbondage Numbers

A set of S vertices in a graph G is a total restrained dominating set if every vertex is adjacent to a vertex in S and every vertex in $V – S$ is also adjacent to a vertex in $V – S$. The total restrained number of G, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a total restrained dominating set of G. Concerning the study of this concept, the reader is referred to [26].

In 2013, Kulli [24] introduced the total restrained nonbondage number $b_{trn}(G)$ of a graph G with no isolated vertices is the maximum cardinality among all sets of edges $X \subseteq E$ for which $G – X$ has no isolated vertex and $\gamma_{trn}(G-X)=\gamma_{trn}(G)$.

The exact values of $b_{trn}(G)$ for some standard graphs are given below.

Theorem 67. If P_p is a path with $p\geq 2$ vertices, then $t_{trn}(P_p)=0$.

Theorem 68. If C_p is a cycle with $p\geq 3$ vertices, then

$b_{trn}(C_p)$ = 1 if p=1,3 (mod 4), $p\geq 4$,
= 0 otherwise..

Theorem 69. If K_p is a complete graph with $p\geq 3$ vertices, then

$b_{trn}(K_p) = \dfrac{p(p-4)+3}{2}$ if $p=2k+1$, $k\geq 1$,

$= \dfrac{p(p-4)+4}{2}$, if $p=2k+2$, $k\geq 1$.

The total restrained bondage number $b_{tr}(G)$ of a graph G with no isolated vertex is the minimum cardinality among all sets of edges $X\subseteq E$ for which $G – X$ has no isolated vertex and $\gamma_{tr}(G – X) > \gamma_{tr}(G)$. This concept was defined by Jafari et al. in [27].

The next result relates between $b_{tr}(G)$ and $b_{trn}(G)$.

Theorem 70. For any graph G without isolated vertices,

$b_{tr}(G) \le b_{trn}(G)+1.$

3.6 Paired Nonbondage Numbers

A dominating set D of a graph G is a paired dominating set if the induced subgraph $\langle D\rangle$ contains a perfect matching. The paired domination number $\gamma_P(G)$ of G is the minimum cardinality of a paired dominating set of G. Paired domination was introduced by Haynes and Slater in [28]. We note that G has a paired dominating set if and only if $\delta(G) \geq 1$.

We introduce the paired nonbondage number $b_{Pn}(G)$ of G as follows.

The paired nonbondage number $b_{Pn}(G)$ of G with $\delta(G) \geq 1$ is the maximum cardinality among all sets of edges $X\subseteq E$ such that $\delta(G – X)\geq 1$ and $\gamma_P(G – X)= \gamma_P(G)$.

We determine the exact values of the paired nonbondage number for some standard graphs.

Theorem 71. For any integer $k\geq 1$ and $p\geq 2$,

$b_{Pn}(P_p)$ = 0 if p=2, 3 or 4,
= k, if $p=4k+j$, $1 \le j \le 4$.

Theorem 72. For any integer $k \geq 1$ and $p \geq 3$,

$b_{Pn}(C_p)$ = 1 if p= 3 or 4,
= k+1 if $p=4k+j$, $1 \le j \le 4$, $p\neq 5$.

Theorem 73. For $p\geq 3$,

$b_{Pn}(K_p) = \dfrac{(p-1)(p-2)}{2}.$

Theorem 74 For a complete bipartite graph $K_{m,n}$, $2\leq m\leq n$,

$b_{p_n}(K_{m,n}) = mn-m-n+1.$

Theorem 75. For a wheel W_p with $p\geq 4$,

$b_{pn}(W_P) = p – 1$.

Let S_k be a a tree obtained by subdividing all edges of a star $K_{1,\,k+1}$, as shown figure 1. It is easy to

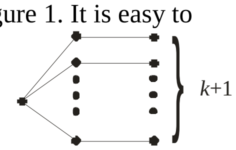

Figure 1

verify that $b_{Pn}(S_k)=k$. We state this fact as the following theorem.

Theorem 76. For any positive integer, k, there exists a tree T with $b_{Pn}(T)=k$.

In [29], Reczek defined the paired bondage number $bp(G)$ of G with $\delta(G) \geq 1$ to be the minimum cardinality among all sets of edges $X\subseteq E$ such that $\delta(G-X)\geq 1$ and $\gamma_P(G-X) > \gamma_P(G)$.

We obtain a relation between the paired bondage number and the paired nonbondage number.

Theorem 77. For any graph G without isolated vertices,

$b_P(G) \le b_{Pn}(G)+1.$

The cycle C_4 achieves this bound.

3.7 Double Nonbondage Numbers

Each vertex of graph G is said to dominate every vertex in its closed neighborhood. A set S of vertices in a graph G is a double dominating set if each vertex in V is dominated by at least two vertices in S. The double domination number $dd(G)$ of G is the minimum cardinality of a double dominating set of G. This concept was defined by Harary et al. in [30].

In [31], Kulli introduced the concept of the double nonbondage number as follows.

The double nonbondage number $b_{dn}(G)$ of G without isolated vertices is the maximum cardinality among all sets of edges $X\subseteq E$ such that $G – X$ has no isolated vertex and $dd(G – X) = dd(G)$.

The exact values of $b_{dn}(G)$ for some standard graphs are given below.

Theorem 78. If C_p is a cycle with $p \geq 3$ vertices, then

$b_{dn}(C_p)$ = 1 if p = 3k+2, $k \geq 1$,
= 0, otherwise.

Theorem 79. If K_p is a complete graph with $p\geq 3$ vertices, then

$$b_{dn}(K_p)=\frac{p(p-5)}{2}+3, \quad \text{if } p\geq 3.$$

Theorem 80. If $K_{m,n}$ is a complete bipartite graph with $3\leq m\leq n$, then

$$b_{dn}(K_{m,n})=mn-2(m+n)+6,$$

Theorem 81. Let G be a graph without isolated vertices. If H is a subgraph of G, then

$b_{dn}(H) \leq b_{dn}(G)$.

The double bondage number $b_d(G)$ of a graph G without isolated vertices is the minimum cardinality among all sets of edges $X\subseteq E$ for which $G - X$ has no isolated vertex and $dd(G - X)>dd(G)$. This concept was introduced by Yogeesh and Soner in [32].

The next result relates $b_d(G)$ and $b_{dn}(G)$.

Theorem 82. For any graph G without isolated vertices,

$$b_{dn}(G)\leq b_{dn}(G)+1.$$

3.8 Edge Nonbondage Numbers

A set of F of edges in a graph $G = (V, E)$ is called an edge dominating set if every edge in $E - F$ is adjacent to at least one edge in F. The edge domination number $\gamma'(G)$ of G is the minimum cardinality of an edge dominating set of G. The concept was defined by Mitchell and Hedetniemi in [33]. For references on this domination in graphs see [34, 35, 36].

In [37] Kulli introduced the concept of the edge nonbondage number as follows.

The edge nonbondage number $b'_n(G)$ of G with no isolated vertex is the maximum cardinality among all sets of edges $X\subseteq E$ such that $G - X$ has no isolated vertices and $\gamma'(G - X)= \gamma'(G)$. If such subset X does not exist in G, then we say that $b'_n(G) = 0$.

The exact values of $b'_n(G)$ for some standard graphs are given below.

Theorem 83. If P_p is a path with $p\geq 2$ vertices, then

$$b'_n(P_p)=0 \quad \text{if } p = 2, 3.$$
$$=\left\lceil\frac{p-1}{3}\right\rceil-1 \quad \text{if } p\geq 4.$$

Theorem 84. If C_p is a cycle with $p\geq 3$ vertices, then

$$b'_n(C_p)=\left\lfloor\frac{p}{3}\right\rfloor.$$

Theorem 85. If K_p is a complete graph with $p\geq 3$ vertices, then

$$b'_n(K_p)=\frac{p(p-1)}{2}-\left\lceil\frac{p}{2}\right\rceil.$$

Theorem 86. If $K_{m,n}$ is a complete bipartite graph, then

$$b'_n(K_{m,n})=0 \quad \text{if } 1 = m \leq n$$
$$= mn - n \quad \text{if } 2 \leq m \leq n.$$

Theorem 87. If W_p is a wheel with $p\geq 4$ vertices, then

$$b'_n(W_p)= p.$$

The edge bondage number $b'(G)$ of a graph G without isolated vertices is the minimum cardinality among all sets of edges $X\subseteq E$ such that $G - X$ has no isolated vertices and $\gamma'(G-X)>\gamma'(G)$. This concept was defined by Kulli in [37]. A relation between the edge bondage number and the edge nonbondage number of G is given below.

Theorem 88. If G is a graph without isolated vertices, then

$$b'(G)\leq b'_n(G)+1.$$

The cycle C_6 achieves this bound.

3.9 Total Edge Nonbondage Numbers

A set F of edges is a total edge dominating set of G if every edge in G is adjacent to at least one edge in F. The total edge domination number $\gamma'_t(G)$ of G is the minimum cardinality of a total edge dominating set of G. The concept of total edge domination was introduced by Kulli and Patwari in [38] and further studied in [39, 40]. In this section, we consider graphs without isolated vertices.

The total edge nonbondage number $b'_{tn}(G)$ of G without isolated vertices is the maximum cardinality among all sets of edges $X \subseteq E$ such that $G - X$ contains no isolated vertices and $\gamma'_t(G - X) = \gamma'_t(G)$. This concept was introduced by Kulli in [41].

The exact values of $b'_{tn}(G)$ for some standard graphs are given below.

Theorem 89. If P_p is a path $p\geq 3$ vertices and $k\geq 1$, then

$b'_{tn}(P_p)=k - 1$, if $p = 4k - 2 + j$, $1 \leq j \leq 4$.

Theorem 90. If C_p is a cycle with $p\geq 3$ vertices and $k\geq 1$, then

$b'_t(C_p)=k$, if $p = 4k - 2 + j$, $0 \leq j \leq 3$.

The total edge bondage number $b'_t(G)$ of G without isolated vertices is the minimum cardinality among all sets of edges $X\subseteq E$ such that $G - X$ contains no isolated vertices and $\gamma'_t(G-X)>\gamma'_t(G)$. This concept was introduced by Kulli in [41].

Theorem 91. For any graph G without isolated vertices,

$$b'_t(G)\leq b'_{tn}(G)+1.$$

3.10 Nonbondage Numbers of Digraphs

3.10.1 Nonbondage Numbers

Let $D = (V, A)$ be a finite directed graph without loops and multiple arcs (but pairs of opposite arcs are allowed). For basic terminology, we refer to Chartrand and Lesnaik [42].

For any vertex $u \in V$ in a digaraph D, the sets $O(u)=\{v/(u,v)\in A\}$ and $I(u)=\{v/(v,u)\in A\}$ are called the outset and inset of u respectively. The indegree and outdegee of u are defined by $id(u) = |I(u)|$ and $od(u) = |O(u)|$. The maximum outdegree of D is denoted by $\Delta^+(D)$.

A set S of vertices in a digraph $D = (V, A)$ is a dominating set if for every vertex $u \in V - S$, there exists a vertex $v \in S$, such that $(v, u) \in A$. The domination number $\gamma(D)$ of D is the minimum cardinality of a dominating set of D.

To minimize the direct communication links in a directed network or a digraph, in [43] Kulli introduced the concept of the nonbondage number in digraphs as follows:

The nonbondage number $b_n(D)$ of a nonempty digraph D is the maximum cardinality among all subsets of arcs $X \subseteq A$ such that $\gamma(D - X) = \gamma(D)$. In the case that there is no such subset X, we put $b_n(D) = 0$.
Kulli [43] determined $b_n(D)$ for some special digraphs.

Proposition 92 [43]. Let $K_{1,p}$ be a directed star in which $od(u)=p$ and $id(u_i)=1$, $1\leq i \leq p$. Then
$$b_n\left(K_{1,p}\right)=0.$$
Proposition 93 [43]. Let $K_{1,p}$ be a directed star in which $od(u_i)=1$, $1\leq i \leq p$ and $id(u)=p$. Then
$$b_n\left(K_{1,p}\right)=p-1.$$

Proposition 94 [43]. For a directed path P_p with $p\geq3$ vertices
$$b_n\left(P_p\right)=\frac{p}{2}-1, \quad \text{if } p \text{ is even},$$
$$=\left\lfloor \frac{p}{2} \right\rfloor, \quad \text{if } p \text{ is odd}.$$

Proposition 95 [43]. For a directed cycle C_p with $p\geq3$ vertices,
$$b_n\left(C_p\right)=\frac{p}{2}, \quad \text{if } p \text{ is even},$$
$$=\left\lfloor \frac{p}{2} \right\rfloor-1, \quad \text{if } p \text{ is odd}.$$

The exact value of $b_n(D)$ of D is given below.
Theorem 96 [43]. For any nonempty digraph D with p vertices and q arcs,
$$b_n\left(D\right)=q-p+\gamma\left(D\right).$$

Theorem 97 [43]. For any digaraph D with p vertices and q arcs,
$$b_n\left(D\right)\leq q-\Delta^{+}\left(D\right).$$

Theorem 98 [43]. For any subdigraph H of a digraph D,
$$b_n\left(H\right)\leq b_n\left(D\right).$$

Corollary 99 [43]. If D has a Hamiltonian circuit, then
$$b_n\left(D\right)\geq\left\lceil \frac{p}{2} \right\rceil.$$

The concept of bondage in digraphs was studied by Carlson and Develin in [44] and by Huang and Xu in [45].
The bondage number $b(D)$ of a digraph D is the minimum cardinality of a set X of edges such that $\gamma\left(D-X\right)>\gamma\left(D\right)$ if such a subset $X \subseteq A$ exists.

The following result gives a relation between the bondage number and the nonbondage number of a digraph.
Theorem 100 [43]. For any digraph D,
$$b\left(D\right)\leq b_n\left(D\right)+1.$$

Corollary 101 [43]. For any digraph D,
$$b\left(D\right)\leq q-\Delta^{+}\left(D\right)+1.$$

3.10.2 Total Nonbondage Numbers
Let $D = (V, A)$ be a digraph in which $id(v) + od(v) > 0$ for all $v\in V$. A subset S of V is called a total dominating set of D if S is a dominating set of D and the induced subgraph $\langle S\rangle$ has no isolated vertices. The total domination number $\gamma_t(D)$ of D is the minimum cardinality of a total dominating set of D, see [46]
The concept of the total nonbondage in digraph was introduced by Kulli in [43] as follows.
The total nonbondage number $b_{tn}(D)$ of a digaraph D without isolated vertices is the maximum cardinality among all subsets of arcs $X\subseteq A$ such that $D - X$ has o isolated vertices and $\gamma_t(D - X)= \gamma_t(D)$.
If $b_{tn}(D)$ does not exist, then we put $b_{tn}(D) = 0$.

Proposition 102[43]. If $K_{1,p}$ is a directed star, then $b_{tn}(K_{1,p})=0$.

Proposition 103[43]. For a directed path P_p,
$$b_{tn}\left(P_p\right)=0, \quad \text{if } p=2, 3, 4,$$
$$=\left\lceil \frac{p}{2} \right\rceil-2, \quad \text{if } p\geq5.$$

Proposition 104 [43]. For a directed cycle C_p with $p\geq3$ vertices,
$$b_{tn}\left(C_p\right)=\left\lfloor \frac{p}{3} \right\rfloor.$$
The concept of total bondage in digraphs was studied by Huang and Xu in [47]. Like undirected graphs, they defined the total bondage number $b_t(D)$ of a digraph D.
We obtain a relation between $b_t(D)$ and $b_{tn}(D)$.

Theorem 105. If D is digraph without isolated vertices, then
$$b_t\left(D\right)\leq b_{tn}\left(D\right)+1.$$

Conclusion
We observe that the study of the nonbondage number of a graph is an important parameter of graphs. This parameter depends entirely on the well known parameter the domination number. Similar definitions can be given for any other parameter of graphs, digraphs, fuzzy graphs, signed graphs and hypergraphs.

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