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A Weighted Bregman Algorithm Applied to Sparse Image Reconstruction

[I]Chun Guan, [II]Boyu Tao
[I,II]College of Electronic Engg., Chongqing University of Posts and Telecomm., Chongqing, China

Abstract
Considering the image detail loss and staircase effect problems caused by the fixed parameters of total variation regularization constraints in image spare reconstruction, this paper proposes a weighted Bregman Algorithm applied to sparse image by restricted with second-order total generalized variation(TGV) model. In this Algorithm, the second-order total generalized variation model is applied to balance the first and second derivative in images, and at the same time, this algorithm can modify the weights on the basis of each iteration solution and tensor function by using the Split Bregman Algorithm. Compared with the TV model and fixed TGV model, simulation results show that this algorithm can both maintain image detail information and image outline, as well as improving PSNR and SSIM of the reconstructed image.

Keywords
Image Processing; Compressed Sensing; Total Generalized Variation; Weighted Regularization Term; Split Bregman Algorithm

I. Introduction

Compressed sensing (CS) theory points out that signal with sparse feature in the complete orthogonal space can be accurately restored by appropriate reconstruction algorithm with high probability, which only needs a few incoherent under-sampling data[1]. In the present stage reconstruction algorithm, the total variation (TV) norm minimization model, which can suppress the noise as well as preserving the image geometric features by combining the sparse image features, has been widely used in sparse image reconstruction[2]. However, the total variation norm only takes advantage of the first derivative information in an image, and it can't reconstruct the texture detail region nicely, where would be easily produce block staircase effect.

Kristian proposed the total generalized variation (TGV) mathematical model in 2010. The basic idea is to introduce high-order gradient information to approximates the objective function by any order polynomial, which effectively solves the staircase problem[3]. Knoll firstly pioneered the generalized variational function to reconstruct the magnetic resonance image in 2011. Compared with total variational constraint, it can significantly improve the reconstructed image artifacts and have a good suppression effect on noise, but the reconstructed image still has high frequency aliasing and jagged edges[4]. Ferstl used the anisotropy total generalized variational function and combined the first-order primitive dual algorithm to reconstruct the deep under-sampled image in 2013[5]. This method needs to obtain the reference data set in advance and optimize it by multiple iterations. Ryan reconstructed the image by projecting image into a normalized interval space and reconstructed it by combining the norm constraint in 2015, which had a nice reconstruction result in noise images. However, this method needs to project all the image pixels, and the difference between the grain size of the projection interval would affect the image reconstruction result [6].

The weight coefficient of the traditional total generalized variational model is a global fixed parameter, which couldn't reconstructed the details and texture regions nicely, as well as leading to the imperfect details in the reconstructed image. Besides, the ordinary way to solve the underdetermined equation is using conjugate gradient algorithm, which has high computational complexity, and takes long time to solve the equation. Thus, this paper proposes a weighted Bregman Algorithm for image sparse reconstruction by combining two order total generalized variation constrain. Experimental results show that this proposed method can improve the linear partial loss of image and improve the preserving ability of edge texture detail in reconstructed image, which actually has some advantages compared than fixed parameter model.

1. Total Generalized Variational Constraints

The total generalized variation model can approach the minimization equation by means of higher order polynomials, and effectively avoid the staircase effect in the total variation regularization term. The total generalized variation model belongs to the Panama space semi norm, which has the characteristics of mapping rotation invariance, lower semi continuous and convex function, and guarantees the integrity of sparse signal mapping. Since the two-dimensional image can be approximated by piecewise affine surface, the two order generalized variational model is used as the sparse constraint condition of the optimization equation:

$$\mathrm{TGV}_a^2(u) = \min_u a_1 \|Du - v\|_1 + a_0 \|\varepsilon(v)\|_1 . \qquad (1)$$

In the above equation, Du is the discrete gradient operator, and v is the two level symmetric tensor function, and $\varepsilon(v)$ is the two order symmetric gradient operator, and a_0 , a_1 is the weight coefficient that is used to balance the first order derivative and the second order derivative of the image. Taking this equation as the regular constraint in the optimization equation, the sparse image reconstruction problem based on compressed sensing is represented as follows:

$$\min_u \frac{\lambda}{2}\|A^{cs}u - y\|_2^2 + \mathrm{TGV}_a^2(u) . \qquad (2)$$

In the equation (2), the $\|\cdot\|_2^2$ represents the second norm in the two-dimensional image space, and y is the sampling results, and u is the input of the original image, and $A^{cs} = \Phi\Psi^T$ is the compressed sensing operator, and $\mathrm{TGV}_a^2(u)$ is the total generalized variation fitting term. In this equation, the first part polynomial is the data fitting term requirements, which means the square error between reconstructed image and original image should be as small as possible thus making ensure that the reconstructed solution should be as close as the exact answer. The second part polynomial, namely the two order TGV norm, is the constrained optimization, which guarantee the noise will not enlarge the reconstruction error or overlay the human magnified. λ is worked as regularization parameter to balance these two polynomials.

II. Weighted Split Bregman Algorithm

The split Bregman iterative algorithm decomposes the original optimization problem into some sub problems that are easily solved and self-closed respectively. By introducing auxiliary variables, this algorithm uses the idea of alternating iteration to solve the sub problems[7]. Compared with the ordinary conjugate gradient method, this algorithm can quickly solve the optimization problem with multiple l_1 norm, and reduce the complexity of equation. Besides, it can quickly converge to stable solution.

In equation (2), the traditional way to solve equation is to set $a_1 = 1$ and $a_0 = 2$. However, when introducing the higher order variational function in the total generalized variational model, this way may lead to less closed matrix multiplication, which means the fixed parameter model cannot be a good trade-off and minimize fitting constraints and affects the linear part and the edge part of the reconstructed image[8].

According to the l_1 norm unconstrained optimization problem, the larger weight coefficient, which means has the greater the penalty value, would have stronger constraint effect in equation. Based on the characteristics of the penalty function, this paper presents a method to dynamically adjust the penalty value. Namely, when the image area is smooth and flat, which has smaller discrete gradient value and tensor function value, the equation should increase the weight value to suppress the staircase effect. And in the texture region, which has larger discrete gradient value and tensor function value, the equation need to reduce weight to reduce the penalty value. In this paper, when calculating the equation (2) in each iteration, a_1 is set as the reciprocal of the reconstruction solution and the l_1 norm of the tensor function, and a_0 is set as the reciprocal of the l_1 norm of the two order symmetric gradient operator in the current iteration step.

After modifying the weighted parameter, the equation (2) can be transformed into the following:

$$\min_{u,v} \frac{\lambda}{2}\left\|A^{cs}u - y\right\|_2^2 + \left(\sum_{i=1}^{n} a_{1i}^k \left\|(Du - v)_i\right\|_1 + \sum_{i=1}^{n} a_{0i}^k \left\|\varepsilon(\mathrm{v})_i\right\|_1\right) \tag{3}$$

$$a_{1i}^k = \frac{1}{\left\|(Du^k - v^k)_i\right\|_1 + \xi\eta_0^k} \tag{4}$$

$$a_{0i}^k = \frac{2}{\left\|\varepsilon(v^k)_i\right\|_1 + \xi\eta_1^k} \tag{5}$$

In this equation, k represents the current iteration number, $\xi\eta_0^k$ and $\xi\eta_1^k$ represent the correction quantity that prevents the equation from being divided by zero, η_0^k and η_1^k represents the average density of the whole space respectively, and n representing the total iteration number, and $\|\square\|_2^2$ is the second norm in the two-dimensional space of image.

To solving the equation (3), the intermediate auxiliary variable $M_i \to (D_u - v)_i$ and $N_i \to \varepsilon(v)_i$ are introduced, and the equation (3) can be converted to the following augmented Lagrange function problem:

$$\begin{aligned} \underset{(u,v,M,N)}{L} &= \frac{\lambda}{2}\left\|A^{cs}u - y\right\|_2^2 + \sum_{i=1}^{n} a_1^i \left\|M_i\right\|_1 + \sum_{i=1}^{n} a_0^i \left\|N_i\right\|_1 \\ &+ \mu_1 \left\|M - (Du - v) - d\right\|_2^2 \\ &+ \mu_2 \left\|N - \varepsilon(v) - b\right\|_2^2 \end{aligned} \tag{6}$$

As in the above equation, every auxiliary variable can be solved as following:

$$M^{k+1} = \underset{M}{\operatorname{argmin}}\left\{\sum_{i=1}^{n} a_{1i}^k \left\|M_i\right\|_1 + \frac{\mu}{2}\left\|M - (Du^k - v^k) - d^k\right\|_2^2\right\},$$

$$N^{k+1} = \underset{N}{\operatorname{argmin}}\left\{\sum_{i=1}^{n} a_{0i}^k \left\|N_i\right\|_1 + \frac{\mu_2}{2}\left\|N - \varepsilon(v^k) - b^k\right\|_2^2\right\},$$

$$\begin{aligned} (u^{k+1}, v^{k+1}) = \underset{(u,v)}{\operatorname{argmin}}\Big\{&\frac{\lambda}{2}\left\|A^{cs}u - y\right\|_2^2 \\ &+ \frac{\mu_1}{2}\left\|M^{k+1} - (Du - v) - d^k\right\|_2^2 \\ &+ \frac{\mu_2}{2}\left\|N^{k+1} - \varepsilon(v^k) - b^k\right\|_2^2\Big\} \end{aligned}$$

$$d^{k+1} = d^k + (Du^{k+1} - v^{k+1} - M^{k+1}),$$

$$b^{k+1} = b^k + (\varepsilon(v)^{k+1} - N^{k+1}). \tag{7}$$

In equation (6) and (7), L is the objective function to be sought, μ_1 and μ_2 are the nonnegative penalty coefficient, M and N are the Lagrange multiplier term, d and b the auxiliary variable. It is pointed out in the literature [9] that equation (6) and equation (7) have the same solution while the auxiliary variable tending to zero in the equation (7).

(1). Sub-Problem Solving Method

According to the expression of each intermediate variable in equation (7), the horizontal discrete gradient and the vertical discrete gradient of the image are calculated respectively, and the horizontal and vertical gradient of the objective function is obtained by the first-order necessary condition and periodic boundary condition. The sub-problem of u^{k+1} can be obtained by fast Fourier transform as following:

$$u^{k+1} = \mathcal{F}^{-1}\left(\frac{\mathcal{F}(\lambda A^{cs*}y + \mu_1 D_x^T(v_x^k + M_x^{k+1} - d_x^k)}{\mathcal{F}(\lambda A^{cs*}A^{cs}u + \mu_1 D_x^T D_x u + \mu_1 D_y^T D_y u)} + \frac{\mu_1 D_y^T(v_y^k + M_y^{k+1} - d_y^k))}{\mathcal{F}(\lambda A^{cs*}A^{cs}u + \mu_1 D_x^T D_x u + \mu_1 D_y^T D_y u)}\right) \tag{8}$$

In the above equation, $\mathcal{F}$ represents the Fourier transform, and

$\mathcal{F}^{-1}$ represents the inverse of Fourier transform, and A^{cs*} represents the conjugate matrix of the squeezed perceptual operator. And through the two-dimensional fast Fourier transform, the problem of (v_x^{k+1}, v_y^{k+1}) in equation (7) are solved as following:

$$\begin{cases} v_x^{k+1} = \mathcal{F}^{-1}\left[\dfrac{\mathcal{F}(A_1).*\mathcal{F}(B_1^k) - \mathcal{F}(A_2).*\mathcal{F}(B_2^k)}{\mathcal{F}(A_1).*\mathcal{F}(A_1) - \mathcal{F}(A_2^T).*\mathcal{F}(A_2)}\right] \\ v_y^{k+1} = \mathcal{F}^{-1}\left[\dfrac{\mathcal{F}(A_1).*\mathcal{F}(B_2^k) - \mathcal{F}(A_2^T).*\mathcal{F}(B_1^k)}{\mathcal{F}(A_1).*\mathcal{F}(A_1) - \mathcal{F}(A_2^T).*\mathcal{F}(A_2)}\right] \end{cases} \quad (9)$$

In the above equation, $\mathcal{F}$ represents of Fourier transform, and $\mathcal{F}^{-1}$ represents inverse Fourier transform, and $.*$ represents matrix point multiplication, and A_1, A_2, B_1, B_2 in equation (10) represents the intermediate variables, i.e.:

$$A_1 = \mu_1 I + \mu_2 D_x^T D_x + \mu_2 D_y^T D_y,\ A_2 = \frac{\mu_2}{2} D_y^T D_x v_y,$$

$$B_1^k = \mu_2 D_1^T (N_{xx}^{k+1} - b_{xx}^k) + \mu_2 D_2^T (N_{xy}^{k+1} - b_{xy}^k) + \mu_1 (D_x u - M_x^{k+1} + d_x^k),$$

$$B_2^k = \mu_2 D_y^T (N_{yy}^{k+1} - b_{yy}^k) + \mu_2 D_x^T (N_{xy}^{k+1} - b_{xy}^k) + \mu_1 (D_y u - M_y^{k+1} + d_y^k). \quad (10)$$

The nonlinear shrinkage contraction operator can be used to solve the M^{k+1} and N^{k+1} problem:

$$\begin{cases} M^{k+1} = \text{shrink}(Du^k(s) - v^k(s) + d^k, \dfrac{a_1^k(s)}{\mu_1}) \\ N^{k+1} = \text{shrink}(\varepsilon(v^k)(s) + b^k(s), \dfrac{a_0^k(s)}{\mu_2}) \end{cases} \quad (11)$$

In the above equation, the shrink$(\cdot)$ is a nonlinear contraction , and .* is the matrix multiplication, and sgn is the symbol function, i.e.:

$$\text{shrink}(u, v) = \max(|u| - v, 0).*\text{sgn}(u). \quad (12)$$

III. Experiment Results and Analysis

The split Bregman algorithm described in equation (6) is used to solve the optimization problem, and the specific algorithm is as follows:

Step 1: initialization parameter in equation (6): a_1^0=1 , a_0^0=2 , u^0, v^0, d^0, b^0, $M^0 = [M_x^0, M_y^0]$=0 , $N^0 = [N_{xx}^0, N_{xy}^0, N_{yy}^0]$=0

Step 2 : execute equation (8) to update u^{k+1}, execute equation (9) to update v^{k+1}, execute equation (11) to update M^{k+1}, N^{k+1}, execute equation (8) to update auxiliary variables d^{k+1}, b^{k+1};

Step 3: according to step 2, then execute equation (4) and (5) to update the weight coefficient a_{1i}^{k+1} and a_{0i}^{k+1};

Step 4: the model parameters are updated continually until the relative mean square error of the image is less than the convergence parameter $\varepsilon = \|u^k - u^{k-1}\|_2^2 / \|u^k\|_2^2$. otherwise, continue to run step 2.

In order to verify the algorithm is effective to maintain the detail features of reconstructed images, this experiment selects two different single channel grayscale image from the source image library in Berkeley. The experimental software platform Matlab 2016a, and the hardware platform is based on a desktop computer with Intel E3-1231 V3 3.4GHz CPU, 16GB memory and windows 10 operation system. Peak Signal to Ratio (RSNR) and Structure Similarity (SSIM) are used to measure the performance of algorithm.

In contrast experiment, the TGV model and the traditional TV model parameters are set according to the literature [8-9]. In this paper, the parameters involved in the algorithm are selected according to the scale traversal method, and the parameter values are set up after several experiments. In the equation (6), this experiment set $\mu_1 = 10^4, \mu_2 = 10^9$, $\lambda = 10^6$, which can balance the generalized error and numeric error. In equation (4) and (5), this experiment set these initial parameters as follows: $\xi = 10^{-4}$, $a_1^0 = 1$, $a_0^0 = 2$. In equation (8), this experiment set u^0 as the original image and v^0 as the initial tensor function, and all of the intermediate variables of initial value are set to 0, while the termination conditions for iterative simulation is setting to $\varepsilon = 10^{-4}$.

(1). Experiment Analysis

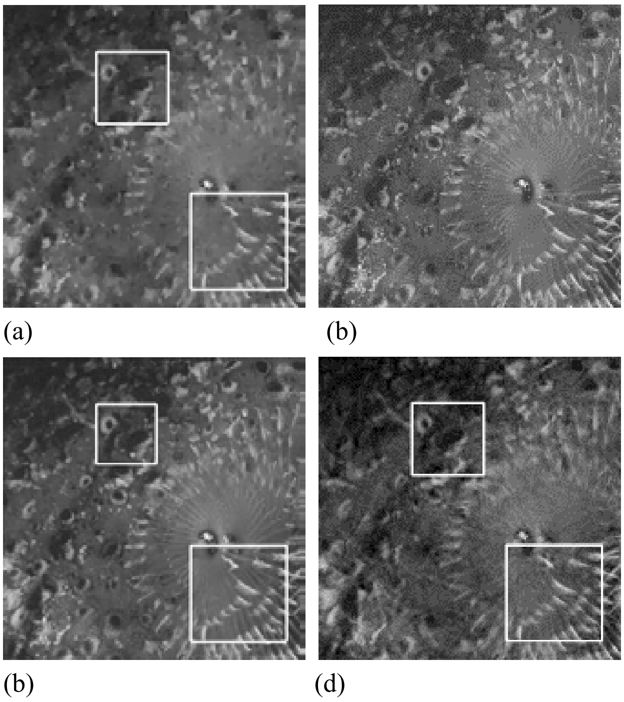
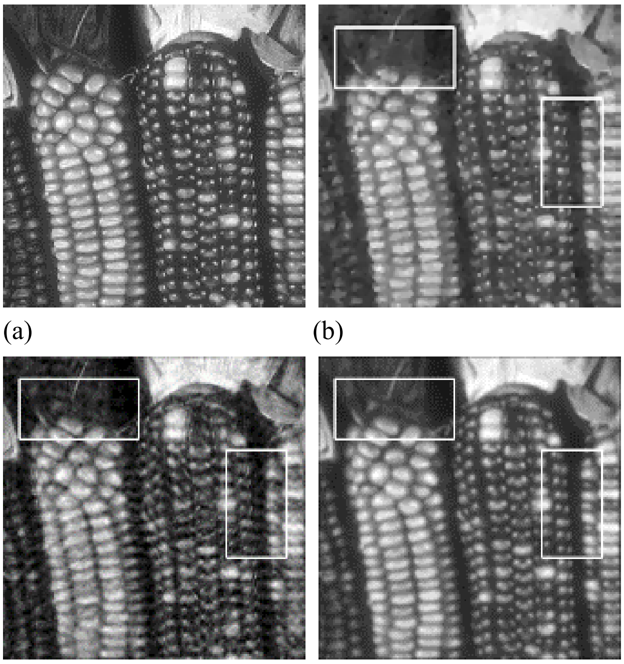


Fig.1 Reconstructed Coral Ticket by different algorithm where sampling ratio is 20%

Figure. 1 is the reconstruction result of three algorithms on the Coral Thicket image where the sampling rate is 20%. Figure 1(a) represents the original image. Figure 1(b) is reconstruct by TV model, which presents an overall image texture details missing and massive staircase effect; Figure 1 (c) is reconstruct by the TGV model, which has improvements in image details, but the reconstruct coral's edge and rhizomes have poor texture; Figure 1 (d) is reconstruct by this paper algorithm, which has more complete overall texture detail. The PSNR improves 1.44dB than the fixed parameter model, and SSIM value increased by 0.041.

Figure 2 is the reconstruction result of three algorithms on a Corn image where the sampling rate of 25%. Figure 2(a) represents the original image. Figure 2 (b) represents the TV model reconstruction result, which has quite obvious staircase effect, and the contours of the corn grain and the edge of the corn leaf are blurred; Figure 2 (c) represents the fixed TGV model reconstruction result. The image detail shows some improvement, and the whole corn and leaf outline are more clearly, but we can still the high frequency blur on corn edge; Figure 2 (d) is reconstruct by this paper algorithm. The corn edge and leaf is relatively completed. Compared to the fixed TGV model, the PSNR value has improved 1.51dB, and SSIM value increased by 0.093.



(a) (b) (c) (d)

Fig.2 Reconstructed Corn image by different algorithm where sampling ratio is 25%

IV. Conclusions

In this paper, a weighted Bregman Algorithm has been proposed for sparse image reconstruction. This algorithm, based on the compressed sensing theory, makes use of the second order total generalized variation constraints and solving the equation with weighted split Bregman Algorithm. This algorithm can adaptively modify the weighting coefficients in the iterative calculation and improve the texture detail retention ability of the reconstructed image. Compared with the fixed parameter model, the experimental results show that this algorithm can improve the reconstructed image quality. However, the reconstruction effect of texture method is still need to be strengthened. And in the furfure, it can be improved by combining with image block theory and Bayesian statistical theory.

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